Last time: inverses $AA^{-1} = I_n = A^{-1}A$

for an $n\times n$ matrix A (they don't always exist)

$\cdot\ (AB)^{-1} = B^{-1}A^{-1}$ $\quad\cdot\ (A^T)^{-1} = (A^{-1})^T$ $\quad\cdot\ (A^{-1})^{-1} = A$

Elementary matrices:

(give rise to Gaussian elimination)

$$D_i^{(\lambda)} = \begin{array}{c} \\ i \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$S_{ij} = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} \ddots & & & & \\ & 0 & \cdots & 1 & \\ & & \ddots & & \\ & 1 & \cdots & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$T_{21}^{(\lambda)} \cdot A$$

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21}+\lambda a_{11} & a_{22}+\lambda a_{12} & \cdots \end{pmatrix}$$

$$T_{ji}^{(\lambda)} = \begin{array}{c} \\ \\ \\ j \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & \lambda & & \ddots & \\ & & & & 1 \end{pmatrix}$$

A square matrix is either not invertible (fewer pivots than its size) or Gaussian elimination produces $A^{-1} = M_k \cdots M_1$, where M_i's are elementary matrices

Today: Gaussian elimination can be phrased in terms of multiplication with elementary matrices even for non-square matrices (just that $\nexists$ inverses)

$$\underset{A}{\begin{pmatrix} 0 & 0 & 2 & 6 \\ 1 & 3 & -1 & 4 \end{pmatrix}} \xrightarrow[\text{and row 1}]{\text{swap row 2}} \underset{S_{12}A}{\begin{pmatrix} \boxed{1} & 3 & -1 & 4 \\ 0 & 0 & 2 & 6 \end{pmatrix}}$$

$$\underset{S_{12}A}{\begin{pmatrix} \boxed{1} & 3 & -1 & 4 \\ 0 & 0 & 2 & 6 \end{pmatrix}} \xrightarrow[\text{row 2 by } \frac{1}{2}]{\text{multiply}} \underset{D_2^{(\frac{1}{2})} S_{12} A}{\begin{pmatrix} \boxed{1} & 3 & -1 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{pmatrix}}$$

$$\underset{D_2^{(\frac{1}{2})} S_{12} A}{\begin{pmatrix} \boxed{1} & 3 & -1 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{pmatrix}} \xrightarrow[\text{to row 1}]{\text{add row 2}} \underset{T_{12}^{(1)} D_2^{(\frac{1}{2})} S_{12} A = REF(A)}{\begin{pmatrix} \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & \boxed{1} & 3 \end{pmatrix}}$$

So
$$A = S_{12}^{-1} \left(D_2^{(\frac{1}{2})}\right)^{-1} \left(T_{12}^{(1)}\right)^{-1} \cdot REF(A)$$
$$= S_{12}\, D_2^{(2)}\, T_{12}^{(-1)} \cdot REF(A)$$

Equivalent characterizations of when a square matrix $A \in \mathbb{R}^{n \times n}$ is invertible

1. A is a product of elementary matrices

$\Updownarrow$ b/c $A = M_k \cdots M_2 M_1 \,\mathrm{REF}(A)$

② $\mathrm{REF}(A)$ has n pivots $\Leftrightarrow$ $\mathrm{REF}(A) = \begin{pmatrix} \boxed{1} & & & & 0 \\ & \boxed{1} & & & \\ & & \boxed{1} & & \\ & & & \boxed{1} & \\ 0 & & & & \boxed{1} \end{pmatrix}$

③ $Ax = 0$ has $x = 0$ as its only solution

$\Updownarrow$ Thm in Lecture 6

pivots = # columns

$\Updownarrow$ Thm in Lecture 6

④ the columns of A are linearly independent

Thm in Lecture 6 $\Updownarrow$ ③

⑤ $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(x) = Ax$ is injective

⑥ $Ax = b$ has at most one solution $x \in \mathbb{R}^n$, $\forall\, b \in \mathbb{R}^n$
(exactly)

$\Updownarrow$ Thm in Lecture 6

pivots = # rows $\Leftrightarrow$ ②

$\Updownarrow$ Thm in Lecture 6

⑦ the columns of A span $\mathbb{R}^n$

$\Updownarrow$ Thm in Lecture 3

⑧ $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(x) = Ax$ is surjective

⑤ & ⑧

$\Updownarrow$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(x) = Ax$ is bijective

(8.5) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = Ax$ is bijective

(9) $\exists\, C \in \mathbb{R}^{n\times n}$ s.t. $CA = I_n$

(7)

$A = M_1 \cdots\cdots M_k$

pick $C = M_k^{-1} \cdots\cdots M_1^{-1}$

(10) $\exists\, C \in \mathbb{R}^{n\times n}$ s.t. $AC = I_n$

(11) A^T is invertible, because $A^T (A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$

(4) (11)

(12) the rows of A are linearly independent

rows of A are columns of A^T

(7) (11)

(13) the rows of A span $\mathbb{R}^n$

New topic: Column operations on $A \in \mathbb{R}^{m\times n}$

1) multiply j-th column by λ : $A \rightsquigarrow A \cdot D_j^{(\lambda)}$

2) swap columns i and j : $A \rightsquigarrow A \cdot S_{ij}$

3) add $\lambda \cdot$ column i to column j : $A \rightsquigarrow A\, T_{ij}^{(\lambda)}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \end{pmatrix} \rightsquigarrow A \cdot T_{12}^{(\lambda)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & \lambda a_{11} + a_{12} \\ a_{21} & \lambda a_{21} + a_{22} \\ a_{31} & \lambda a_{31} + a_{32} \\ \vdots & \end{pmatrix}$$

$$T_{23}^{(\lambda)} = \begin{pmatrix} 1 & & & \\ & 1 & \lambda & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

adds $\lambda \cdot col_2$ to col_3

adds $\lambda \cdot row_3$ to row_2

$$T_{32}^{(\lambda)} = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & \lambda & 1 & \\ & & & 1 \end{pmatrix}$$

adds $\lambda \cdot col_3$ to col_2

adds $\lambda \cdot row_2$ to row_3

Please fill out indicative evaluation

New topic: Determinant (of a square matrix)

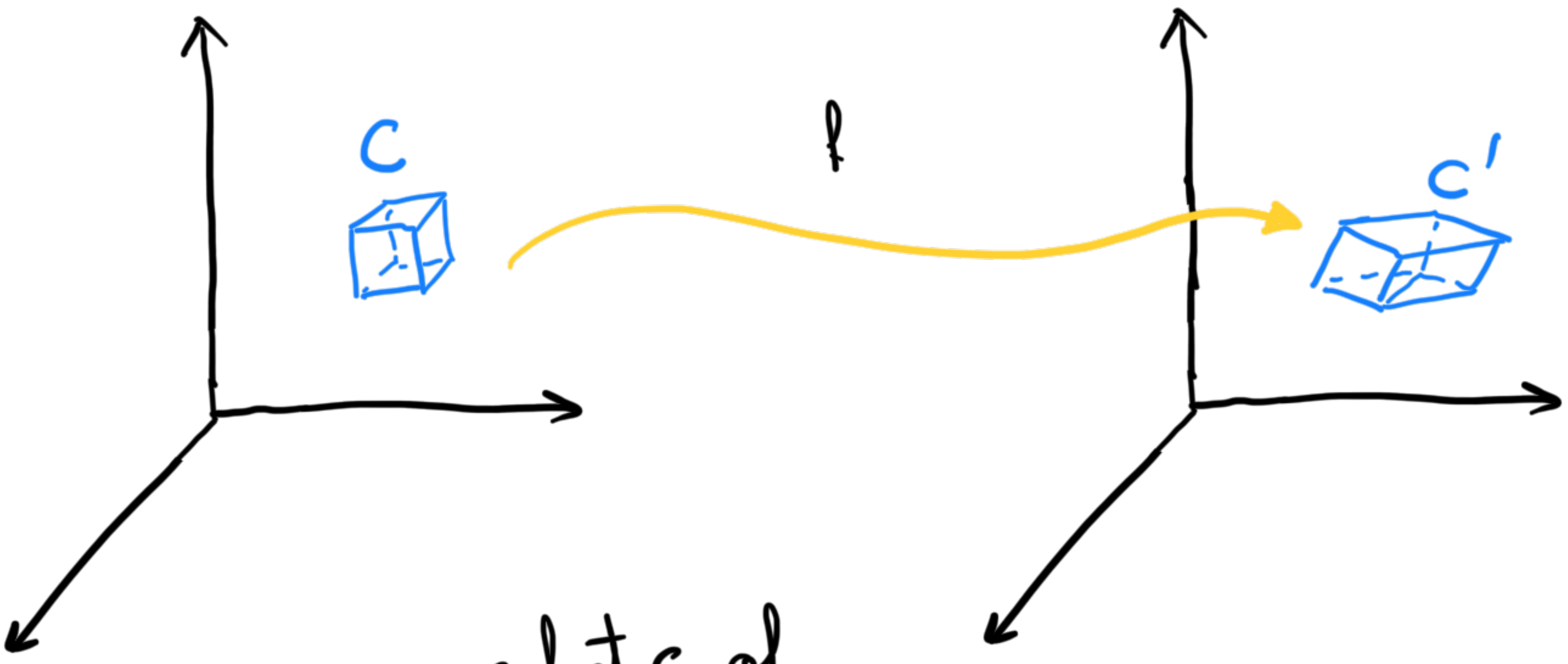
Motivation: $\det(A) \in \mathbb{R}$ tells us if A is invertible or not

THM 10.1: $\det(A) \neq 0 \iff A$ invertible

$$\det(a) = a$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

DEF 10.2 $A \in \mathbb{R}^{n \times n} \rightsquigarrow f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\qquad\qquad\qquad\qquad\qquad\quad x \longrightarrow Ax$



pick any subset C of
and look at $C' = f(C)$

does not depend on choice of C

Let $\boxed{\det(A) = \pm \dfrac{\mathrm{Vol}(c')}{\mathrm{Vol}(c)}}$, i.e. the factor by which f increases/decreases $\underbrace{\text{volumes}}$

we'll explain this sign later

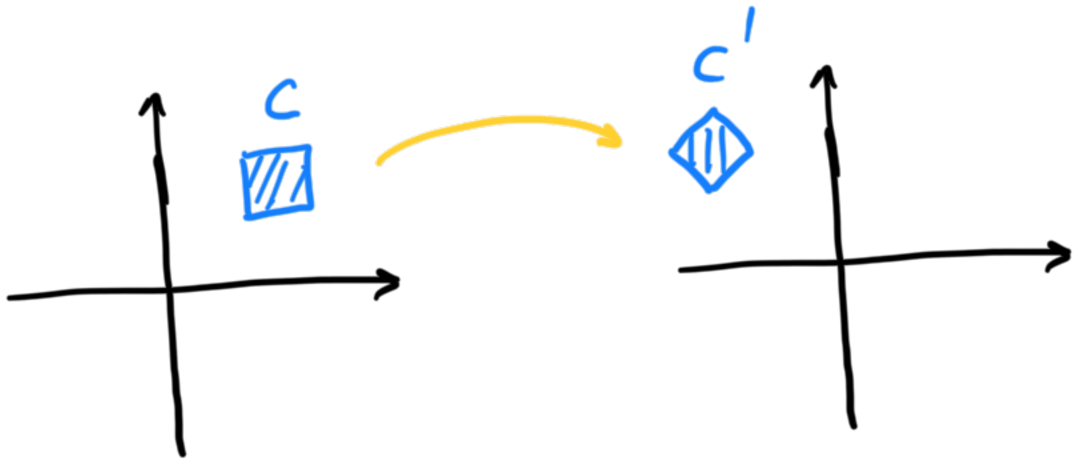
- length for $\mathbb{R}^1$
- area for $\mathbb{R}^2$
- volume for $\mathbb{R}^3$
- "hypervolume" for $\mathbb{R}^{\geq 4}$

Examples for $n=2$:

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, rotation by α $\rightsquigarrow$ $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

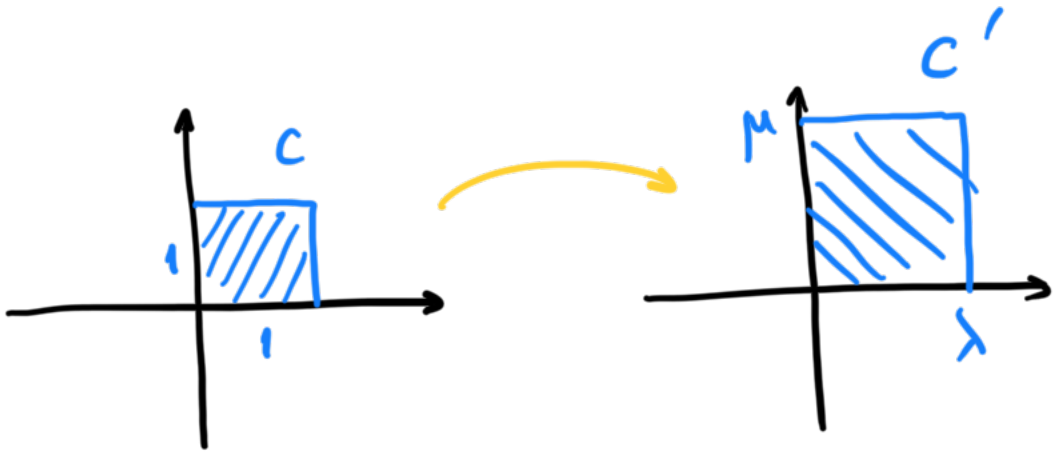
$\Downarrow$

$\det(A) = (\cos\alpha)(\cos\alpha) - (-\sin\alpha)(\sin\alpha) = 1$



$\det(A) = \dfrac{\mathrm{Area}(c')}{\mathrm{Area}(c)} = 1$ (rotation doesn't change areas)

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, rescaling $\rightsquigarrow$ $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $\Rightarrow \det(A) = \lambda\mu$

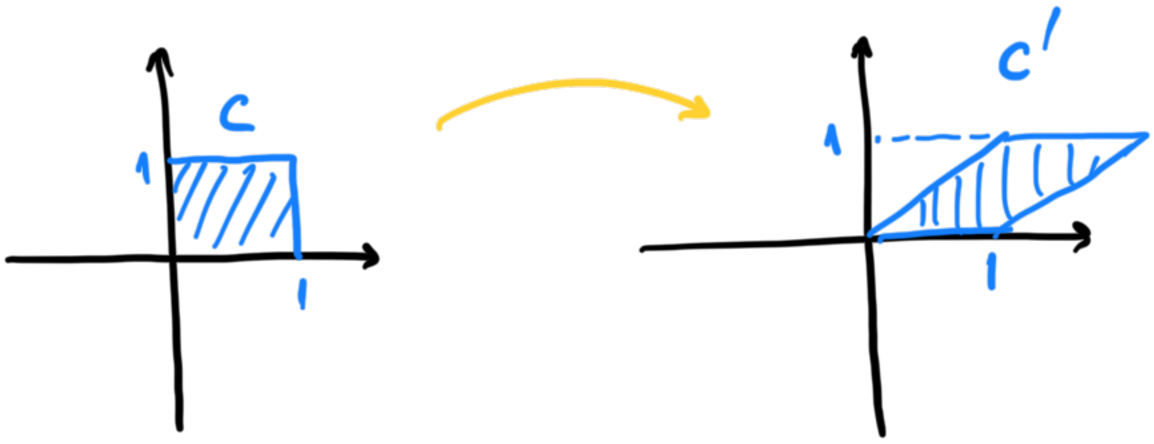


$\det(A) = \dfrac{\mathrm{Area}(c')}{\mathrm{Area}(c)} = \lambda\mu$ (rescaling rescales areas by $\lambda\mu$)

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ shearing $\rightsquigarrow A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

$\Downarrow$

$\det(A) = 1 \cdot 1 - 0 \cdot \lambda = 1$

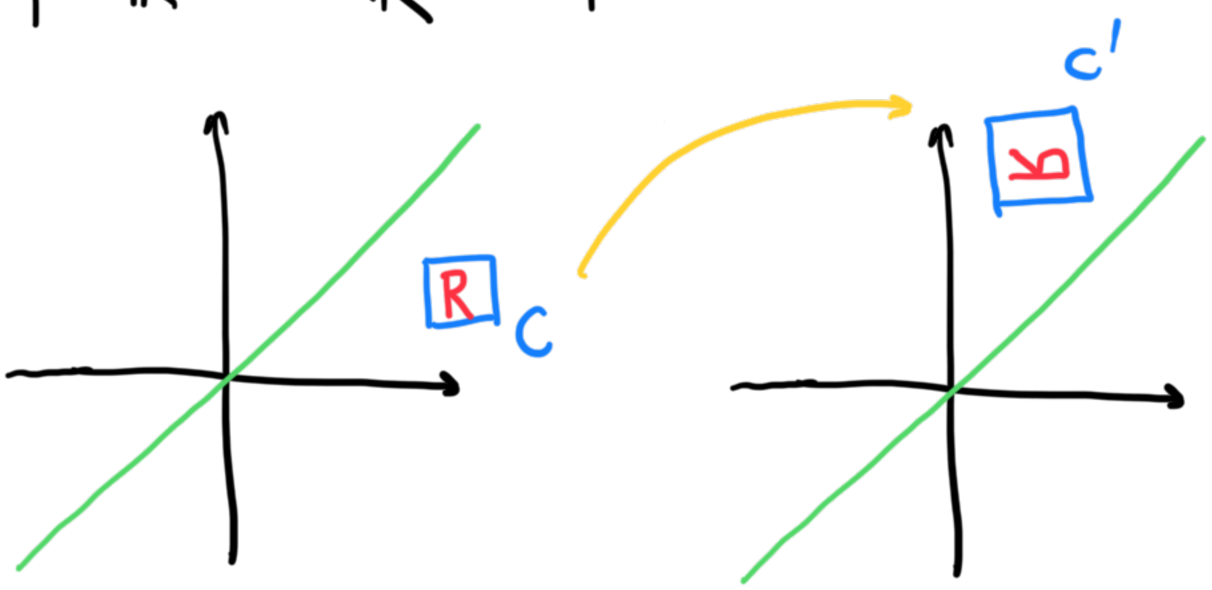


$$\det(A) = \frac{\text{Area}(c')}{\text{Area}(c)} = 1$$ (shearing doesn't change area)

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection $\rightsquigarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Downarrow$

$\det(A) = 0 \cdot 0 - 1 \cdot 1 = -1$



$$\det(A) = \frac{\text{Area}(c')}{\text{Area}(c)} = -1$$, because reflection changes orientation like a mirror (R → Я)

THM 10.3: $\det(AB) = \det(A)\det(B)$

Proof: let f and g be the linear functions corresponding to A and B

$$f \circ g: \mathbb{R}^n \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$$

$$x \longrightarrow Bx$$

$$Y \longrightarrow AY$$

$$z \longrightarrow ABz$$

$$\text{subset } C \xrightarrow{g} C' \xrightarrow{f} C''$$

by definition of determinants, we have

$$\left.\begin{aligned} \det(A) &= \frac{Vol(C'')}{Vol(C')} \\ \det(B) &= \frac{Vol(C')}{Vol(C)} \end{aligned}\right\} \Rightarrow \det(A)\det(B) = \frac{Vol(C'')}{Vol(C)} = \det(AB)$$

Recall: $A = \underbrace{M_1 \cdots\cdots M_K}_{\text{elementary matrices}} \cdot \underbrace{REF(A)}_{= I_n \Leftarrow A \text{ is invertible}}$

$\det(A) = \det(M_1) \cdots \det(M_K)$ means we can compute dets by Gaussian elimination

$$\det\left(D_i^{(\lambda)}\right) = \det\begin{pmatrix} \ddots & & 0 \\ & \lambda & \\ 0 & & \ddots \end{pmatrix} = \lambda \quad \text{(rescaling)}$$

$$\det\left(S_{ij}\right) = \det\begin{pmatrix} \ddots & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & \ddots \end{pmatrix} = -1 \quad \text{(reflection)}$$

$$\det\left(T_{ij}^{(\lambda)}\right) = \det\begin{pmatrix} \ddots & & \lambda \\ & \ddots & \end{pmatrix} = 1 \quad \text{(shearing)}$$

$$\det\left(T_{ij}\right) = \det\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix} = 1$$

$$\Longrightarrow \det(A) = (-1)^{\#\text{row swaps}} \cdot \text{product of all row rescaling factors}$$

$$A = D_i^{(3)}\, T_{jK}^{(5)}\, S_{LM}\, D_N^{(7)} \cdots\cdots$$

$$\det(A) = 3 \cdot 1 \cdot (-1) \cdot 7 \cdots\cdots$$

Example from last time:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{pmatrix} = S_{13}\, T_{21}^{(2)}\, D_2^{(-1)}\, T_{12}^{(3)}\, T_{23}^{(1)}\, T_{13}^{(-2)}$$

$$\det(A) = -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 1 = 1$$

Prop: • $\det(I_n) = 1$ (identity function doesn't rescale)

• $\det(A^{-1}) = \dfrac{1}{\det A}$ (inverse function dilates by inverse amount)

• $\det\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & & a_n \end{pmatrix} = \det\left(D_1^{(a_1)} D_2^{(a_2)} \cdots D_n^{(a_n)}\right) = \det\left(D_1^{(a_1)}\right) \cdots \det\left(D_n^{(a_n)}\right) = a_1 a_2 \cdots a_n$

• $\det\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = \det\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix} = a_1 \cdots a_n$

- $\det(A) = \det(A^T)$

$$\cdots\; S_{ij}\; D_i^{(\lambda)}\; T_{ij}^{(\lambda)} \;\cdots \qquad\qquad \cdots\; T_{ji}^{(\lambda)}\; D_i^{(\lambda)}\; S_{ij} \;\cdots$$

$$S_{ij} \to -1, \quad D_i^{(\lambda)} \to \lambda, \quad T_{ij}^{(\lambda)} \to 1 \qquad\qquad T_{ji}^{(\lambda)} \to 1, \quad D_i^{(\lambda)} \to \lambda, \quad S_{ij} \to -1$$

(because D and S matrices are symmetric, they are unchanged by transposing)

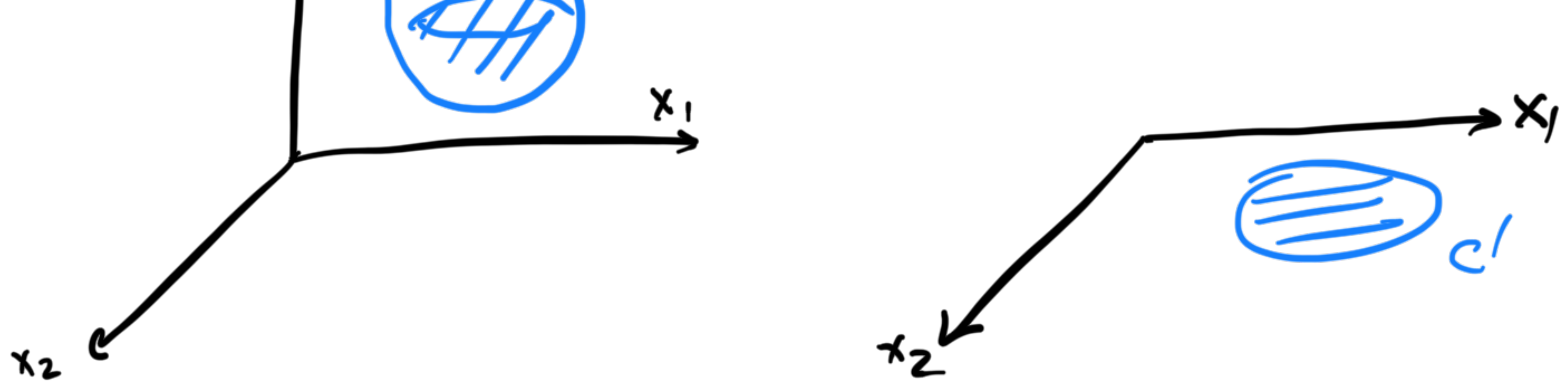
What happens if A is not invertible, i.e.

$$A = M_1 \cdots\cdots M_K \begin{pmatrix} & * & \\ 0 & 0\,0\,0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\det(A) = \text{stuff} \cdot \det\begin{pmatrix} * \\ 0\,0\,0\,0 \end{pmatrix} = \text{stuff} \cdot 0 = 0$$

full row of zeroes means that the corresponding function squashes the last dimension



and so $\det\begin{pmatrix} * \\ 0\;0\;0\;0 \end{pmatrix} = \dfrac{\mathrm{vol}(\quad)}{\mathrm{vol}(\quad)} = \dfrac{0}{\text{non-zero}} = 0$

Next time : determinants by cofactor expansion

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \dots + (-1)^{n-1} a_{1n}\det(A_{1n})$$

where A_{ij} = matrix A with row i and column j removed